Reliability of Randomly Excited Structures

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A method is developed for the prediction of the reliability of narrow-band structures under stationary random excitations. The present approach takes into account the interaction of catastrophic failure modes and fatigue failure modes as well as the statistical variation of the material strength. Fracture mechanics and extreme point processes are employed throughout the formulation. The effect of loading history on the structural reliability is accounted for, which, however, cannot be accomplished using the cumulative damage hypothesis and the Palmgren-miner rule. It is demonstrated that neglecting the interactions of failure modes, or disregarding the statistical dispersion of the material strength results in an unconservative reliability estimate. This tends to become more critical as the flaw propagation factor or the dispersion of the material strength increases.

I. Introduction

STRUCTURAL reliability under random excitations is a well recognized problem in aerospace engineering. In-flight measurements taken on a number of recent space flights[1] show that spacecraft excitations and responses contain to a large degree steady-state random components in addition to certain deterministic phenomena. Aircraft structural loads and excitations that are a result of engine noise and boundary-layer turbulence are usually also classified as stationary random. The necessity of studying structural reliability and failure under stationary random excitations is therefore apparent.

Because of mathematical expedience, two essential modes of structural failure are usually investigated separately as follows. 1) Catastrophic failure occurs instantaneously because some response measures, such as displacements or stresses, exceed their limiting values. This type of failure is referred to as the first passage failure, and the associated probability of catastrophic failure is called the first passage or the first excursion probability. Although the exact solution for a simple structural system excited by white noise has not been obtained,[2] many approximate solutions have been proposed,[3–7] 2) Fatigue failure occurs because of the successive incremental reduction of the limiting response measures as a result of flaw growth with each load repetition. Fatigue failure estimations are usually made using the well-known Palmgren-Miner rule[8,9] or using the fracture mechanics approach.[10]

Thus far, in random vibration, the material properties of structures have not been considered as random variables, and the relevancy of structural reliability to optimum structural design has not been investigated thoroughly. On the other hand, structural reliability under static or quasi-static random loadings, taking into account statistical variations of material properties, has been investigated quite extensively.[11,12] The relevancy of structural reliability to optimum structural design, coupled with proof testing and structural strength deterioration (fatigue and creep), has also been investigated.[13–16]

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In this paper, the concepts of fracture mechanics and the point processes associated with stationary narrow-band random processes[6] are employed to estimate the reliability of narrow-band structures under stationary random excitations. Two failure modes are allowed for, i.e., catastrophic and fatigue failures, and the statistical variations of the material properties are taken into account. A quantitative investigation is conducted to determine the interaction of the failure modes and the relevancy to structural reliability estimates. The premises of this study are as follows: 1) catastrophic failure occurs as soon as the stress response process exceeds the critical fracture stress (threshold level) of the structure; and 2) successive incremental reduction of the critical fracture stress (threshold level), as a result of stress repetition, increases the catastrophic failure rate in time. Therefore, the problem treated here is a first passage problem with a monotonically decreasing probabilistic barrier level that depends not only on the dispersion of the material strength, but also on the history of the random response process.

II. Strength Deterioration of Structures

One way of explaining structural failure mechanisms and processes, such as fatigue failures, is by the use of the concepts of fracture mechanics.[16,17] The mechanisms of fatigue failure can then be summarized as follows: 1) flaw initiation; 2) flaw propagation; and 3) catastrophic failure. The last two mechanisms of the fatigue process are of primary concern in structural analysis and design. The flaw initiation stage is the one about which little is known at present. It can usually be assumed, however, that the material already has flaws that propagate under repetitive stress applications with sufficiently high amplitude. The fatigue failure process is thus described by the growth of flaws in a structure until the applied stress at any flawed point exceeds the critical fracture stress associated with the flaw at that critical point and catastrophic failure occurs. Flaw propagation under random loadings has been investigated to some extent in Refs. 18 and 19.

Let A represent the flaw size at the critical point of the structure. According to the well-known Griffith-Irwin equation,

$$A = Q(K_{IC}/R)^2 \tag{1}$$

$$A = Q(K_I/S)^2 \tag{2}$$

where K_{IC} is the critical stress intensity factor, R is the critical fracture stress or the resisting stress associated with the flaw size A, K_I is the stress intensity factor associated with the flaw size A and the applied stress S, and Q is a state parameter.[16–20]

The flaw propagation law takes the form[10,18–21]

$$dA/dn = CK^b{}_I \tag{3}$$

stating that the rate of flaw extension with respect to the number of stress cycles n is proportional to the bth power of the stress intensity factor K_I, where C is a suitable constant. By theoretical analyses as well as direct measurements of flaw propagations[10,20,21] it has been shown that b ranges from 2 to 4 depending on the material used and environmental conditions such as temperature, corrosion, etc.

Substituting Eq. (2) into Eq. (3), one obtains

$$dA/dn = KS^bA^{b/2} \tag{4}$$

where $K = C/Q^{b/2}$ is a constant.

In the following approach, it is assumed, for convenience and without essential loss of generality, that $b = 2$; it will be shown later that this applies also to the cases $b = 3$ or 4. Note that b is held arbitrary for S which is a parameter in Eq. (4).

Integrating Eq. (4) successively with respect to each cycle and summing, one obtains

$$\ln A_n - \ln A_0 = K \sum_{j=1}^{n} S^b{}_j \tag{5}$$

where S_j is the jth peak of the stress response $S(t)$. The initial flaw size A_0, and the flaw size after n stress cycles, A_n, are respectively related to the initial resisting stress R_0 and the resisting stress after n stress cycles R_n through Eq. (1). Hence, substituting Eq. (1) into Eq. (5) yields

$$R_n = R_0 \exp(-Z_n) \text{ if the event } \left\{ \bigcap_{j=1}^{n} (R_{j-1} > S_j) \right\} \text{ occurs} \tag{6}$$

$$Z_n = \frac{K}{2} \sum_{j=1}^{n} S_j{}^b \tag{7}$$

Equation 6 holds only under the condition that $R_{j-1} > S_j$ for $j = 1,2,\ldots,n$, i.e., the structure has survived n cycles of stress application, since otherwise structural failure would have occurred before the nth stress cycle and R_n becomes meaningless. Therefore, R_n is a conditional random variable. It is clear from Eqs. (6) and (7) that the structural resisting stress after n stress cycles R_n decreases monotonically with respect to the number of stress cycles n.

A possible event of structural strength deterioration given in Eqs. (6) and (7) is shown schematically in Fig. 1 in which R_0 and S_j; $j = 1, 2 \ldots n$, are random variables. It follows then from Eqs. (6) and (7) that the statistical distribution of R_n depends on the statistical distributions of R_0 as well as the stress peaks S_j; $j = 1,2,\ldots,n$. Therefore, the statistical characteristics of the stress peaks, instead of the random response process $S(t)$, are of primary interest. The statistical characteristics of stress peaks S_j; $j = 1,2,\ldots n$, which can be derived from the stress response process, $S(t)$, and the extreme point process[6] will be discussed briefly.

III. Extreme Point Processes

Consider $S(t)$ a stationary narrow-band Gaussian stress response process with mean zero and a mean square spectral density, $\phi(\omega)$. Let ω_0 be central frequency of $\phi(\omega)$ and T_0 the period of $S(t)$, i.e., $T_0 = 2\pi/\omega_0$. If S_m and $\bar{S}_m$ denote, respectively, the m-th local maximum (peak) and the m-th local minimum (trough) of $S(t)$, then the local maxima form a stationary point process called the maximum point process, $\{S_n\}$, and the local minima also form a stationary minimum point process, $\{\bar{S}_n\}$.[6] Because of the narrow-band characteristics of $S(t)$, both, S_m and $\bar{S}_m$ will fall within the mth cycle of $S(t)$ or within the time interval $[(m-1)T_0, mT_0]$. Now let the extreme point process $\{\eta(n)\}$ of $S(t)$ be the mixed point process consisting of the maximum point process, $\{S_n\}$, and the negative of the minimum point process, $\{-\bar{S}_n\}$. A mixed point process composed of events E is a mixture of two component point processes with events E_1 and E_2 (or events of type 1 and type 2) in which an event of type 1 always follows an event of type 2 and vice versa.[6] Thus, the 2mth and the $2m + 1$th point of the extreme point process, $\{\eta(n)\}$, represent, resepctively, the mth peak, S_m, and the negative of the mth trough, $-\bar{S}_m$, both occuring in the time interval $[(m-1)T_0, mT_0]$, i.e., $\eta(2m) = S_m$ and $\eta(2m+1) = -\bar{S}_m$. Evidently, this implies that $\eta(n)$ represents peak values if n is even, whereas it represents the absolute values of troughs if n is odd. The extreme point process $\{\eta(n)\}$ is again stationary.

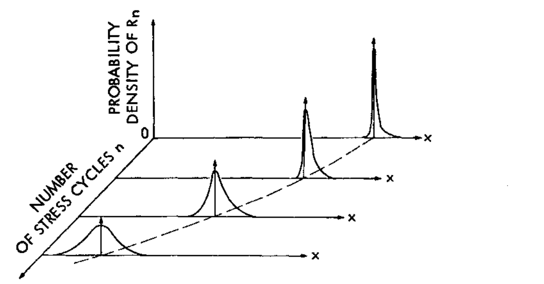

Fig. 1 Probability density function of the structural resisting strength R_n vs stress cycles n.

Thus, the first passage one-sided barrier problem of $S(t)$ can be approximated by the first passage problem of the maximum point process, $\{S_n\}$, in which the time is measured in terms of the discrete number of cycles n. Similarly, the first passage two-sided barrier problem of $S(t)$ with barriers $\pm\lambda(\lambda > 0)$ can be approximated by the one-sided barrier problem of the extreme point process $\{\eta(n)\}$ with a barrier at λ. In this case, however, the time is measured in terms of the number of half cycles.[6]

The distribution function $F_{S_n}(x)$ of the maximum point process $\{S_n\}$ is the Rayleigh distribution

$$F_{S_n}(x) = 1 - \exp\{-x^2/2\sigma_S{}^2\} \tag{8}$$

in which σ_S is the standard deviation of $S(t)$.

Because of the narrow-band characteristics of $S(t)$, the joint density function $f_{S_jS_{j+m}}(x,y)$ of S_j and S_{j+m} spaced at a distance of m cycles can be approximated by the joint density function $f_S(x,y;\tau)$ of the envelope function of $S(t)$ spaced at a distance $\tau = mT_0$,

$$f_{S_jS_{j+m}}(x,y) = f_S(x,y;mT_0) \tag{9}$$

It has been shown in Ref. 22 that

$$f_S(x,y;\tau) = \frac{xy}{\sigma_S{}^4[1 - k_0{}^2(\tau)]} I_0\left[\frac{xyk_0(\tau)}{\sigma_S{}^2[1 - k_0{}^2(\tau)]}\right] \times \exp\left\{\frac{-(x^2+y^2)}{2\sigma_S{}^2[1 - k_0{}^2(\tau)]}\right\} \tag{10}$$

in which $I_0[\cdot]$ is the zero order modified Bessel function of the first kind and

$$k_0(\tau) = [v_1{}^2(\tau) + v_2{}^2(\tau)]^{1/2} \tag{11}$$

$$v_1(\tau) = 2\sigma_S{}^{-2} \int_0^\infty \phi(\omega) \cos(\omega - \omega_0)\tau d\omega \tag{12a}$$

$$v_2(\tau) = 2\sigma_S{}^{-2} \int_0^\infty \phi(\omega) \sin(\omega - \omega_0)\tau d\omega \tag{12b}$$

Equation 10 is based on Rice's definition of an envelope which applies without ambiguity for symmetric spectra. Since, however, narrow-band structural response spectra are not symmetrical, a suitable substitution for the central frequency ω_0 should be made. For a single-degree-of-freedom system excited by white noise, three possible substitutions for approximation may be used[5,8]; 1) natural frequency ω_n, 2) damped natural frequency $\omega_d = \omega_n(1 - \zeta^2)^{1/2}$, and 3) centroid frequency

$$\omega_c = \int_0^\infty \omega\phi(\omega)d\omega \Big/ \int_0^\infty \phi(\omega)d\omega = (\omega_n/\omega_d) \times \{1 - \tan^{-1}[2\zeta(1 - \zeta^2)^{1/2}/(1 - 2\zeta^2)]/\pi\},$$

where ζ is the damping coefficient. However, since an appropriate choice for ω_0 may be critical in certain cases[7] and because Eq. 10 will be used later, a careful evaluation of the effect of these approximations on $k_0(\tau)$ and $f_S(x,y;\tau)$ (Eq. 10) at $\tau = mT_0$ has been made. It is found that a particular choice for ω_0 is not critical for $f_S(x,y;mT_0)$. In fact, three different approximations yield practically the same result, when the damping is small. As a result, we chose the natural frequency ω_n for ω_0, for the sake of simplicity, and

$$k_0(mT_0) \approx \rho(mT_0) \approx \exp(-2m\pi\zeta) \tag{12c}$$

where $\rho(\tau)$ is the correlation coefficient of $S(t)$ and $S(t + \tau)$.

IV. Failure Rate and Reliability

The probability that a structure, having survived n stress cycles, will fail in the $n + 1$-th stress cycle is denoted by $h(n)$ and is called failure rate, risk function, or hazard function, ie.,

$$h(n) = P\left[R_n \le S_{n+1} \middle| \bigcap_{j=1}^{n} (R_{j-1} > S_j)\right]; \quad n = 1,2,\ldots \tag{13}$$

$$h(0) = P[R_0 \le S_1]$$

Where R_n is given by Eq. (6). Substitution of Eq. (6) into Eq. (13) yields

$$h(n) = P\left[R_0 \exp(-Z_n) \le S_{n+1} \middle| \bigcap_{j=1}^{n} (R_{j-1} > S_j)\right] \tag{13a}$$

Define a random variable $\mathbf{R}_n$ such that

$$\mathbf{R}_n = R_0 \exp(-Z_n) \tag{13b}$$

Substitution of Eq. (13b) into Eq. (13a) yields

$$h(n) = P\left[\mathbf{R}_n \le S_{n+1} \middle| \bigcap_{j=1}^{n} (R_{j-1} > S_j)\right] \tag{13c}$$

Note that $\mathbf{R}_n$ defined in Eq. (13b) is different from R_n in Eq. 6. The difference lies in the fact that $\mathbf{R}_n$ in Eq. (13b) is an unconditional random variable which holds without the condition that $R_{j-1} > S_j$, for $j = 1,2,\ldots,n$. Hence, $\mathbf{R}_n$ is referred to as the unconditional resisting strength after n stress cycles.

The probability that the structure will survive N stress cycles, denoted by $L(N)$, is called reliability. Successive application of Eq. (13) yields

$$L(N) = \prod_{n=0}^{N-1} [1 - h(n)] \tag{14}$$

and for $h(n) \ll 1$, a conservative good approximation can be obtained as

$$L(N) \simeq \exp\left\{- \sum_{n=0}^{N-1} h(n)\right\} \tag{15}$$

Eq. 13 is the first passage problem with one-sided barrier. The two-sided barrier problem can be accounted for in a similar manner by considering the extreme point process $\{\eta(n)\}$.[6] Two approximate solutions, the Poisson approximation and the clump size approximation, are considered in the following.

A. Poisson Approximation

If the point process $\{S_n\}$ is assumed to be of Poisson type, Ref. 6, i.e., S_{n+1} is independent of S_j; $j = $ i,2,...,n and the event $\{\mathbf{R}_n \le S_{n+1}\}$ is independent of the past event

$$\left\{\bigcap_{j=1}^{n} (R_{j-1} > S_j)\right\}$$

then the Poisson failure rate $h_p(n)$ follows from Eq. 13c as

$$h_p(n) = P[\mathbf{R}_n \le S_{n+1}] = \int_0^\infty [1 - F_{S_{n+1}}(x)]f_{\mathbf{R}_n}(x)dx \tag{16}$$

in which $f_{R_n}(x)$ and $F_{S_{n+1}}(x)$ are, respectively, the probability density function of $\mathbf{R}_n$ and the distribution function of S_{n+1} given in Eq. (8).

Equation (13b) can be written as follows:

$$\ln\mathbf{R}_n = \ln R_0 - Z_n \tag{17}$$

The statistical distribution of the initial resisting stress R_0 is determined by material specimen tests, the form of which is material characteristics and reflects environmental conditions. Experience shows that the normal, lognormal, gamma or Weibull distribution can be applied in most cases.[11,12] Using the lognormal distribution for R_0 because of simplicity and because it is quite reasonable for many engineering materials[11] the term $\ln R_0$ in Eq. 17 is normally distributed.

It is observed from Eq. 7 that Z_n is the sum of dependent random variables S_j^b; $j = 1,2,\ldots,n$. As will be shown later, the distribution of Z_n is asymptotically normal. Therefore, for large n, the distribution of Z_n can reasonably be approximated by the normal distribution. Furthermore, we shall restrict ourself to the case of high-cycle fatigue where the average number of cycles to failure is large. Hence, the effect of Z_n (fatigue failure mode) on the failure rate $h_p(n)$ [see Eqs. (16, 13c and 7)] is negligible for small value of n. Therefore, the distribution of Z_n for small n is irrelevant to the reliability estimate. As a result, the distribution of Z_n is approximated by the normal distribution. It therefore follows from Eq. 17 that $\ln\mathbf{R}_n$ is also normally distributed with the mean value μ_n and the variance σ_n^2 given by:

$$\mu_n = \mu_0 - \mu_z, \quad \sigma_n^2 = \sigma_0^2 + \sigma_z^2 \tag{18}$$

where μ_0 and σ_0^2 are the mean value and the variance, respectively, of the normal random variable $\ln R_0$. The mean value μ_z and the variance σ_z^2 of Z_n can be computed using Eqs. (7, 8 and 10). The results are as follows:

$$\mu_z = \frac{n}{2} K[(2)^{1/2}\sigma_S]^b \,\Gamma\left(1 + \frac{b}{2}\right) \tag{19}$$

$$\sigma_z^2 = \frac{n}{4} K^2 \,[(2)^{1/2}\sigma_S]^{2b}\left[\Gamma(1 + b) - \Gamma^2\left(1 + \frac{b}{2}\right)\right] + \frac{1}{4} K^2[(2)^{1/2}\sigma_S]^{2b}\Gamma^2\left(1 + \frac{b}{2}\right) \times \left\{\sum_{m=1}^{n-1} (n - m)\left[{}_2F_1\left(-\frac{b}{2}, -\frac{b}{2}; 1, \; k_0^2(mT_0)\right) - 1\right]\right\} \tag{20}$$

$$E[S_j^b S^b_{j+m}] = ((2)^{1/2}\sigma_S)^{2b}\Gamma^2\left(1 + \frac{b}{2}\right) \times {}_2F_1\left(-\frac{b}{2}, -\frac{b}{2}; 1, \; k_0^2(mT_0)\right) \tag{21}$$

where $\Gamma(\cdot)$ and ${}_2F_1(\cdot)$ are, respectively, the gamma function and the hypergeometric function, and $E[\cdot]$ denotes the expectation. Further evaluation of σ_z^2 requires a specific choice for the mean square spectral density of the response process $S(t)$ in order to specify the parameter $k_0^2(mT_0)$ and hence the hypergeometric function.

For a single-degree-of-freedom oscillator under white noise excitation, Crandall et. al,[8] obtained the following simplified expressions for σ_z^2 when the damping coefficient ζ is small and when b is odd

$$\sigma_z^2 = \frac{n}{4\zeta} K^2 [(2)^{1/2}\sigma_S]^{2b}\Gamma^2\left(1 + \frac{b}{2}\right) f_1(b) \qquad (22)$$

$$V_z = [f_1(b)/n\zeta]^{1/2} \qquad (23)$$

where V_z is the coefficient of variation of Z_n, and $f_1(b)$ is given in Table 1. It is interesting to note that Shinozuka[23] obtained a similar result, in a somewhat different manner, for b equal to even number. Equation (23) indicates that the statistical dispersion of Z_n diminishes as n increases, and hence the dispersion of $\mathbf{R}_n$, Eq. (13b), depends with increasing n mainly on the dispersion of the initial resisting stress R_0. This indicates the importance of taking into account the statistical variation of the material strength.

Thus the Poisson failure rate $h_p(n)$ follows from Eqs. (8, 16 and 18) as

$$h_p(n) = \int_0^\infty [x\sigma_n (2\pi)^{1/2}]^{-1} \exp\{-[x^2/2\sigma_S^2] - [(\ln x - \mu_n)^2/2\sigma_n^2]\} dx \qquad (24)$$

Once $h_p(n)$ is computed, the reliability $L(N)$ can be obtained from either Eq. (14) or (15) by replacing $h(n)$ by $h_p(n)$.

It should be noted that the particular choice of lognormality for R_0 is not essential here. Any other distribution can be used if there is reason to believe that it represents the material better.

From physical reasoning[4] it is appropriate to assume that the correlation coefficient $\rho(mT_0)$ of the responses $S(t)$ and $S(t + mT_0)$ decreases monotonically to zero with increasing m, i.e., $\rho(m_1T_0) > \rho(m_2T_0)$ for $m_2 > m_1$, and $\rho(mT_0) \rightarrow 0$ as $m \rightarrow \infty$. This assumption is obviously satisfied, for example, by the response process of a single-degree-of-freedom oscillator excited by white noise, as can be observed from Eq. (12c). Therefore, the following important properties of the stationary point process $\{S_n^b\}$, can be obtained from Eqs. (10, 20 and 21),

$$\left.\begin{array}{lll} \sigma_z^2 \sim n\sigma; & \sigma < \infty & \\ E[|S_j^b|^{2+\delta}] < \infty; & \delta > 0 & \\ E[S_j^bS^b_{j+m}] - E[S_j^b]E[S_{j+m}^b] \rightarrow 0 & & \text{as } m \rightarrow \infty \\ E[S_j^bS^b_{j+m}] > E[S_j^bS_{j+l}^b] & & \text{for } l > m \end{array}\right\} \qquad (25)$$

Since Eq. 25 satisfies conditions for a stationary point process $\{S_n^b\}$ to have asymptotic normality,[24,25] i.e., the sum of S_j^b; $j = 1,2,\ldots,n$, tends to be asymptotically normal for increasing n, Z_n is therefore asymptotically normal.

B. Clump Size Approximation

A useful concept was suggested by Lyon,[26] that the level crossings by $S(t)$ are not independent but tend to occur in clumps of dependent crossings. For the narrow-band random process, since each positive upward level crossing by $S(t)$ results in a peak, the events that the peaks, S_j; $j = 1,2,\ldots$, exceeding a threshold level are not independent but tend to occur in clumps. The Poisson assumption implies that one clump of peaks being above the threshold level consists of only one peak (Ref. 6). Hence, if clumps are assumed to occur independently, the average failure rate denoted by $\bar{h}(n)$ follows from Eq. (24) as

$$\bar{h}(n) = \int_0^\infty \{x\sigma_n(2\pi)^{1/2}E[M|x]\}^{-1} \exp\{-[x^2/2\sigma_S^2] - [(\ln x - \mu_n)^2/2\sigma_n^2]\} dx \qquad (26)$$

in which $E[M|x]$ is the average clump size given that the barrier level is equal to x. The structural reliability based on clump size approximation can then be obtained from Eq. (14) by replacing $h(n)$ by $\bar{h}(n)$.

Table 1 Dependence of $f_1(b)$ in Eq. (22) on b

b	$f_1(b)$	b	$f_1(b)$
1	0.0414	9	10.7
3	0.369	11	31.5
5	1.280	13	96.7
7	3.72	15	308

Approximate estimation of average clump size

The exact estimation of the average clump size is rather difficult. However, approximate estimation has been given in Ref. 6, which compares favorably with the simulation results of Ref. 27 as follows:

$$E[M|x] \simeq 1 + \sum_{m=1}^{\infty} m(q_m - q_{m+1})/q_0 \qquad (27)$$

in which

$$q_0 = P[S_j \geq x] = \exp(-x^2/2\sigma_S^2) \qquad (28)$$

$$q_m = P[S_j \geq x, S_{j+m} \geq x] = \int_x^\infty \int_x^\infty f_S(y,z;mT_0)dydz \qquad (29)$$

with $f_S(y,z;mT_0)$ given by Eq. (10). The estimate of $E[M|x]$ in Eq. (27) was originally given in Ref. 28 in which, however, the quantities q_0 and q_m are different from those given in Eqs. (28) and (29) of Ref. 6. The series of Eq. (27) is truncated at the kth term at which $q_k \approx q_0^2$, i.e., $q_k - q_{k+1} \simeq 0$.

In numerical computations of $\bar{h}(n)$ using Eq. (26), it is convenient to obtain first the curves of average clump sizes associated with different barrier levels from Eq. (27), and then to obtain $E[M|x]$ for any value of x by interpolation if necessary. The numerical computations involved in Eqs. (27–29) have been discussed in Ref. 6.

Estimation of average clump size based on available simulation results

For a single-degree-of-freedom oscillator excited by white noise, some numerical simulation results of the mean number of cycles to failure, $\bar{n}$, for different barrier levels, x, and damping coefficients ζ are available in Ref. 27. It has been shown,[6] that according to the principle of maximum entropy, the corresponding average clump size $E[M|x]$ associated with the only information $\bar{n}$ is

$$E[M|x] = \bar{n} \exp(-x^2/2\sigma_S^2) \qquad (30)$$

V. Effect of Loading History

One of the advantages of the flaw propagation approach is that the effect of loading history on strucural reliability is accounted for. Consider, for instance, the initial structural resisting strength R_0 a random variable and the following two loading cases: 1) 5 cycles of high amplitude stresses S followed by 5 cycles of low-amplitude stresses $\tilde{S}$ where $\tilde{S} < S$ and 2) 5 cycles of low amplitude stresses $\tilde{S}$ followed by 5 cycles of high amplitude stress S. According to the cumulative damage hypothesis and the Palmgren-Miner rule, the probability of surviving 10 cycles of stress application for case 1) is equal to that for case 2).[8]

In the present approach, however, it follows from Eqs. (6) and (7) that the structural resisting strengths R_n, $n = 1,2,\ldots$ and the failure rates $h(n)$, $n = 1,2,\ldots$ are different for both different loading cases. It can easily be shown numerically that the structural reliabilities for case 1) and case 2) are different. This predicted phenomenon is completely in agreement with experimental observations.

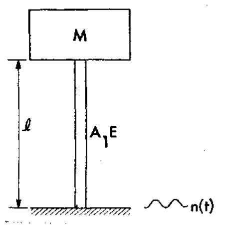

Fig. 2 A simple structure.

VI. Flaw Propagation Law for $b = 3$ or any Number

It has been mentioned in Sec II that, for the sake of simplicity of the presentation, a particular flaw propagation law, using $b = 2$ in Eq. (4), was chosen. If, however, the particular material under consideration, or the environmental conditions encountered are such that $b = 3$, for instance, then, Eq. (4) is integrated successively for each cycle. Summing the parts, one obtains again $\mathbf{R}_n = R_0 - Z_n$, where

$$Z_n = (U/2) \sum_{j=1}^{n} S_j^3,$$

with $U = Q^{1/2}K_{IC}K$, in which Q and K_{IC} are the same as those appearing in Eq. (1).

Recall that the distribution of Z_n can be approximated by the normal distribution, and if R_0 is assumed normally distributed, the distribution function of $\mathbf{R}_n$ is then normal with the mean value $\mu_n = \mu_0 - \mu_z$, and the variance $\sigma_n^2 = \sigma_0^2 + \sigma_z^2$, where μ_0 and σ_0^2 are, respectively, the mean value and the variance of R_0, and μ_z and σ_z^2 can be obtained from Eqs. (19) and (20) with K being replaced by U. Consequently, the Poisson failure rate can be obtained as

$$h_p(n) = [2(2)^{1/2}\sigma_n W]^{-1} \operatorname{erf}(-\mu_n/2\sigma_n^2 W) \times \exp\left\{(-\mu_n^2/2\sigma_n^2)\left[1 + \frac{1}{1 + (\sigma_n^2/\sigma_S^2)}\right]\right\} + \Phi(-\mu_n/\sigma_n) \tag{31}$$

where

$$W = [(2\sigma_S^2)^{-1} + (2\sigma_n^2)^{-1}]^{1/2} \tag{32}$$

and $\Phi(\cdot)$ is the standardized Gaussian distribution function.

The second term of the right hand side of Eq. (31) can usually be neglected, since it is extremely small compared to the first term. The computation of the average failure rate $\bar{h}(n)$ based on the clump size approximation can be obtained in a similar fashion.

For $b = 4$ or any number, similar approach discussed above can be applied to find failure rates $h_p(n)$ and $\bar{h}(n)$ numerically, except that the computational effort may be much involved.

VII. Numerical Example

Making the transformation $\gamma = x/\sigma_S$, the average failure rate $\bar{h}(n)$ given in Eq. (26) can be written as

$$\bar{h}(n) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty \frac{1}{\gamma\sigma_n E[M|\gamma\sigma_S]} \exp\left(-\frac{\gamma^2}{2} - \frac{1}{2}\left\{\frac{\ln[\gamma\ (V_0^2+1)^{1/2}/\nu_0] + \mu_z}{\sigma_n}\right\}^2\right) d\gamma \tag{33}$$

in which $V_0 = [\exp(\sigma_0^2) - 1]^{1/2}$ is the coefficient of variation of R_0, and $\nu_0 = \bar{R}_0/\sigma_S$ is the normalized initial barrier level with $\bar{R}_0$ the mean value of R_0. ν_0 is related to a central safety factor ν_c as, $\nu_c = \bar{R}_0/\mu_S = \nu_0(2/\pi)^{1/2}$, where μ_S is the mean value of S_n. Since ν_c is based on the central measures of location of R_0 and S_n, it is called central safety factor, which plays a major role in the reliability-based optimum structural design, Refs. 13–16. In deriving Eq. (33), the relationship, $\mu_0 = \ln[\bar{R}_0/(V_0^2 + 1)^{1/2}]$, associated with the lognormal distribution of R_0 has been employed.

With the aid of Eqs. (18), (19) and (22), σ_n and μ_z appearing in Eq. (33) can be written as follows:

$$\mu_z = \frac{n}{2} C_p[(2)^{1/2}/\nu_0]^b\Gamma(1 + b/2) \tag{34}$$

$$\sigma_n^2 = \ln(V_0^2 + 1) + \frac{n}{4\zeta} C_p^2\ [(2)^{1/2}/\nu_0]^{2b}\Gamma^2(1 + b/2)f_1(b) \tag{35}$$

where $C_p = K\bar{R}_0^b$ is a flaw propagation factor depending on the particular material and environmental conditions.

It follows from Eqs. (33–35) that the average failure rate and hence the structural reliability are characterized by C_p, V_0 and ν_0, where C_p and V_0 are material properties associated with flaw propagation and initial strength dispersion (or initial flaw size dispersion), and ν_0 is the normalized initial barrier level (or a measure of the central safety factor) relating the applied load characteristics to the structural design.

Consider a simple structure, that can be idealized as a single-degree-of-freedom system, excited by white noise, $n(t)$, where the displacement $Y(t)$ is related to $n(t)$ as follows:

$$\ddot{Y}(t) + 2\zeta\omega_n\dot{Y}(t) + \omega_n^2 Y(t) = n(t) \tag{36}$$

The stress response process $S(t)$ is related to $Y(t)$ by $S(t) = DY(t)$, with D being a suitable constant.

To show the relationship of ν_0, or of the central safety factor, $\nu_c = \nu_0(2/\pi)^{1/2}$, to the structural design and to the applied load characteristics, the simple structure consisting of a mass M supported by a beam with area A_1, length L and Young's modulus of elasticity E is considered, (see Fig. 2). It can easily be shown that ω_n appearing in Eq. (36) is $\omega_n^2 = A_1E/ML$ and $\sigma_S^2 = \phi_0\pi E^2/2\zeta\omega_n^3L^2$, where ϕ_0 is the mean square spectral density of white noise $n(t)$. Hence,

$$\nu_0 = \bar{R}_0 \left(\frac{2\zeta}{\pi\phi_0}\right)^{1/2} \left(\frac{A_1}{M}\right)^{3/4} \left(\frac{L}{E}\right)^{1/4} \tag{37}$$

In the present example, it is assumed that failure is due to tension only. The damping coefficient is assumed 0.02. The average clump size $E[M|\gamma\sigma_S]$ associated with different values of γ is plotted in Fig. 3, where for $\gamma < 4.0$, $E[M|\gamma\sigma_S]$ is obtained from Eq. 30 using simulated $\bar{n}$ of Ref. 27. For the high threshold level, such as $\gamma \geq 4.0$, no simulation result $\bar{n}$ is currently available, because in order to obtain accurate average cycles to failure by simulation a considerable amount of sample functions have to be simulated which sometimes is prohibitive even with a high-speed digital computer. Therefore, $E[M|\gamma\sigma_S]$ is computed using Eqs. 27–29 for $\gamma \geq 4.0$.

Failure rates $h_p(n)$ associated with different values of C_p, V_0 and ν_0 are plotted in Fig. 4. In Fig. 4a, $C_p = 0$, representing the case where the fatigue failure mode is neglected, and hence failure rates are constant. In Figs. 4b and 4c, failure rates are plotted vs the number of stress cycles n. It is clearly shown in Fig. 4 that for this example: 1) because

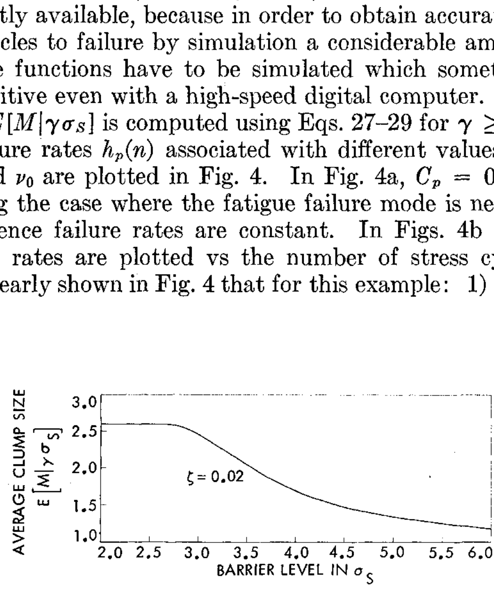

Fig. 3 Average clump size $E[M|\gamma\sigma_S]$ vs barrier level.

of the interaction of the fatigue failure mode, the failure rate increases monotonically with respect to n and C_p; 2) the failure rate increases as the statistical dispersion, V_0, of the initial material strength, R_0, increases; and 3) the effect of the fatigue failure mode is negligible for small n justifying the assumption of normality for the random variable Z_n, Eqs. (7) and (18).

In Fig. 5, the structural reliability, $L(N)$, based on clump size approximation is plotted as a function of stress cycles N. It is observed from Fig. 5 that neglecting the effect of the fatigue failure mode, i.e., $C_p = 0$, results in an unconservative reliability prediction. This situation becomes much more critical as the flaw propagation factor C_p increases. It is further observed that the structural reliability decreases as the statistical dispersion V_0 of the material strength R_0 increases. This reflects an important fact that overlooking the statistical variation of material strength results in an unsafe design.

Although a particular flaw propagation law, $b = 2$, and a particular distribution function (lognormal) for the initial material strength, R_0, are used in the numerical example, above trends of observations are believed to hold for other flaw propagation laws such as $b = 3$ or $b = 4$, as well as other statistical distribution functions for the initial material strength, R_0.

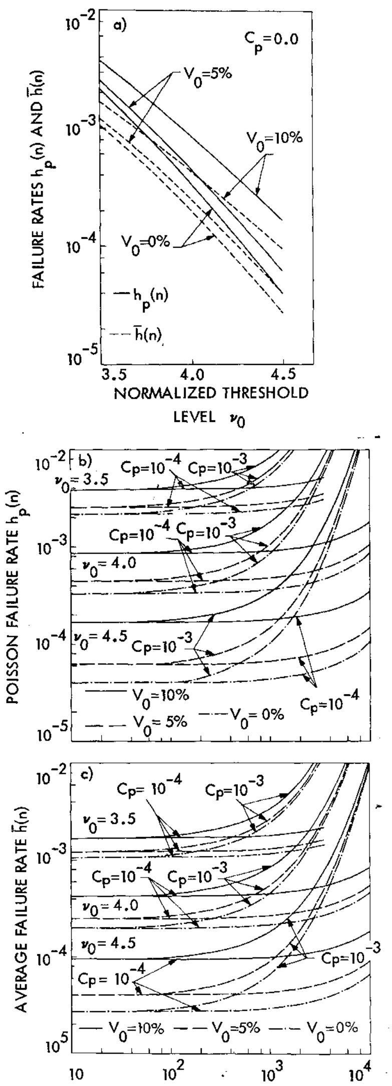

Fig. 4 Failure rates $h_p(n)$ and $\bar{h}(n)$; a) constant failure rates $h_p(n)$ and $\bar{h}(n)$ for $C_p = 0.0$; b) Poisson failure rate $h_p(n)$ vs number of stress cycles n; c) average failure rate $\bar{h}(n)$ based on the clump size approximation vs number of stress cycle n.

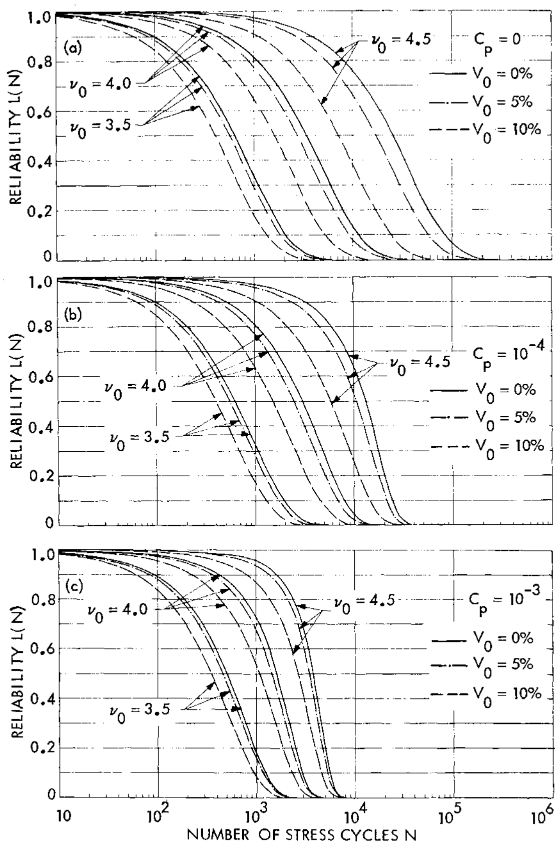

Fig. 5 Structural reliability $L(N)$ vs number of stress cycles N; a) $C_p = 0$, b) $C_p = 10^{-4}$ and c) $C_p = 10^{-3}$.

VIII. Conclusions

A method is developed for the prediction of the reliability of narrow-band structures under stationary random excitations. Two important ideas have been employed to investigate the interaction of catastrophic failure modes and fatigue failure modes: a) catastrophic failure takes place as soon as the stress response process exceeds the critical fracture stress (threshold level) of the structure; and b) successive incremental reduction of the critical fracture stress, as a result of stress repetition, increases in time the catastrophic failure rate. Fracture mechanics and extreme point processes are employed throughout the formulation. The effects of loading history on the structural reliability are accounted for which, however, cannot be accomplished using the cumulative damage hypotheses and the Palmgren-Miner rule. The statistical variation of material strength is also taken into account in estimating the structural reliability. It is demonstrated that neglecting the interactions of failure modes, or disregarding the statistical dispersion of the material strength, results in an unconservative reliability estimate. This tends to become more critical as the flaw propagation factor C_p or the dispersion of the material strength V_0 increases.

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